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Note on the capillary thread instability for fluids of equal viscosities

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An analytic formula is derived for the Rayleigh growth rate of a fluid cylinder immersed in a fluid of equal viscosity, and an extension is given for concentric fluid threads.

The interfacial instability of a fluid cylinder of viscosity $\lambda\mu$ suspended in a fluid of viscosity μ was first studied by Rayleigh (1892) for a viscous fluid in air ($\lambda = \infty$). The usual solution approach for arbitrary λ (e.g. Tomotika 1935; Chandrasekhar 1961) is to assume an axial interface distortion proportional to $e^{\sigma t}\cos kz$, and determine the form of the stream function for low-Reynolds-number motions internal and external to the thread. The application of boundary conditions for the velocity and the tangential and normal stresses leads to an eigenvalue problem in the form of a 4×4 system of equations whose solution yields $\sigma(k;\lambda)$. Simple analytical formulae for $\sigma(k;\lambda)$ are known for $\lambda = 0$ (Tomotika 1935) and $\lambda = \infty$ (Rayleigh 1892).

Here we provide a direct method for treating the intermediate case where $\lambda = 1$, thereby obtaining a formula for the linear growth rate. The idea of a ring forcing is used which directly incorporates in the governing equations the normal stress jump arising from interfacial tension influences. The use of Hankel transforms leads to an explicit expression for the radial velocity which is simply related to the growth rate.

Consider a fluid thread with a nearly circular shape $r = a(1+\epsilon(t)\cos kz)$; $\epsilon \ll 1$. The viscosities are the same in both fluids ($\lambda = 1$). We seek to solve Stokes equations, and the boundary conditions are linearized about $r = a$ (Tomotika 1935). In particular, there is a jump in normal stress equal to $(\gamma\epsilon/a)(1-(ak)^2)\cos kz$ across $r = a$; the jump is represented using the delta function $\delta(r-a)$. The momentum equation, including the presence of the radially directed ring force, and the continuity equation are

$$-\nabla p + \mu\nabla^2 \boldsymbol{u} + \boldsymbol{e}_r\delta(r-a)\frac{\gamma\epsilon}{a}\left(1-(ak)^2\right)\cos kz = \boldsymbol{0} \qquad \text{and} \qquad \nabla\cdot\boldsymbol{u} = 0. \tag{1}$$

Solutions consistent with the sinusoidal shape perturbation have the form

$$(u_r(r,z), u_z(r,z)) = \frac{\gamma\epsilon}{\mu}\left(\bar{u}_r(r)\cos kz, \bar{u}_z(r)\sin kz\right) \qquad p(r,z) = \frac{\gamma\epsilon}{a}\bar{p}(r)\cos kz. \tag{2}$$

The resulting ordinary differential equations for $(\bar{u}_r, \bar{u}_z, \bar{p})$ are conveniently solved by using Hankel transforms, where

$$\mathscr{H}_n\{\phi(r)\} = \int_0^\infty \phi(r) r J_n(sr)\,\mathrm{d}r \equiv \Phi(s), \qquad \mathscr{H}_n^{-1}\{\Phi(s)\} = \int_0^\infty \Phi(s) s J_n(sr)\,\mathrm{d}s \equiv \phi(r), \tag{3}$$

and so we take $(U_r(s), U_z(s), P(s)) = \left(\mathscr{H}_1\{\bar{u}_r\}, \mathscr{H}_0\{\bar{u}_z\}, \mathscr{H}_0\{\bar{p}\}\right)$. Then the continuity equation is $sU_r + kU_z = 0$ and the momentum equation becomes

$$\frac{s}{a}P - \left(s^2 + k^2\right) U_r + \left(1 - (ak)^2\right) J_1(sa) = 0, \tag{4a}$$

$$\frac{k}{a}P - \left(s^2 + k^2\right) U_z = 0. \tag{4b}$$

Solving for $U_r(s)$ and taking the inverse Hankel transform gives

$$u_r(r,z) = \frac{\gamma\epsilon}{\mu}k^2\left(1 - (ak)^2\right)\cos kz \int_0^\infty \frac{sJ_1(sr)J_1(sa)}{\left(s^2+k^2\right)^2}\mathrm{d}s. \tag{5}$$

The interface shape evolves according to $u_r(r = a, z) = a\dot{\epsilon}\cos kz$ from which we see that $\epsilon(t) \propto \exp(\sigma t)$, and using (5) the growth rate σ follows from

$$\sigma = \frac{\gamma}{\mu a}k^2\left(1 - (ak)^2\right)\int_0^\infty \frac{sJ_1^2(sa)}{\left(s^2+k^2\right)^2}\,\mathrm{d}s. \tag{6}$$

The integral may be evaluated (differentiate equation 6.535, Gradshteyn & Ryzhik 1965) and so we arrive at σ as a function of the dimensionless wavenumber ak:

$$\sigma(k) = \frac{\gamma}{\mu a}\left(1 - (ak)^2\right)\left[I_1(ak)K_1(ak) + \frac{ak}{2}\left(I_1(ak)K_0(ak) - I_0(ak)K_1(ak)\right)\right], \tag{7}$$

where the I_i and K_i are modified Bessel functions. Using *Mathematica*, equation (7) may be obtained via simplification of Tomotika's general result with $\lambda = 1$. Equation (7), obtained using the simple derivation above, represents a new analytical formula for this classical fluid dynamics problem.

This solution procedure using a ring forcing also yields the linear growth rate for concentric fluid cylinders of equal viscosity in an infinite fluid. We consider two such cylinders with inner radius a_1 and outer radius a_2, and interfacial tensions γ_1 and γ_2, respectively. Application of the above ideas yields a quadratic equation for σ:

$$\left(\sigma - \frac{k^2\gamma_1}{a_1\mu}(1-(a_1k)^2)A(a_1,a_1)\right)\left(\sigma - \frac{k^2\gamma_2}{a_2\mu}(1-(a_2k)^2)A(a_2,a_2)\right) = \frac{k^4\gamma_1\gamma_2}{a_1a_2\mu^2}(1-(a_1k)^2)(1-(a_2k)^2)A(a_1,a_2)^2, \tag{8}$$

where (equation 6.541, Gradshteyn & Ryzhik 1965)

$$A(b,c) = \int_0^\infty \frac{sJ_1(sb)J_1(sc)}{(s^2+k^2)^2}\,\mathrm{d}s = -\frac{1}{2k}\frac{\mathrm{d}}{\mathrm{d}k}(I_1(bk)K_1(ck)) \quad (b \leqslant c). \tag{9}$$

Inspection of (8) shows that σ is always real and that in the limit $a_2/a_1 \to \infty$ the unstable modes predicted by (8) are the same as those given by (7). Finally, this method, and equation (8), may be generalized to any number of concentric fluid cylinders.

REFERENCES

Chandrasekhar, S. 1961 *Hydrodynamic and Hydromagnetic Stability.* Oxford University Press.

Gradshteyn, I. S. & Ryzhik, I. M. 1965 *Tables of Integrals, Series and Products*, 5th edn, p. 702, Academic.

Rayleigh, Lord 1892 On the instability of a cylinder of viscous liquid under the capillary force. *Phil. Mag.* **34**, 145–154.

Tomotika, S. 1935 On the instability of a cylindrical thread of a viscous liquid surrounded by another viscous fluid. *Proc. R. Soc. Lond.* A **150**, 322–337.